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Logical Quantizations of First-Order Structures

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Many, though surely not all, mathematical structures can successfully be depicted by theories in (infinitary, many-sorted) first-order logic. The principal concern of this paper is to show systematically, following the lines of our previous papers, how to quantize a wide and important class of mathematical structures of first-order description, namely, the class of mathematical structures delineated by so-called limit theories. By way of example, not only every equational theory (e.g., the theory of groups and homomorphisms), but also the theory of partially ordered sets and order-preserving mappings and that of Banach spaces and contractive linear transformations are limit theories, so that they are susceptible of logical quantization.

0. INTRODUCTION

The most tractable mathematical structures are sets endowed with operations subject to certain equations. They are what are called algebraic structures in the narrowest sense, and their general theory was denominated universal algebra, for which the reader is referred to Grätzer (1979). Groups and rings are typical examples of algebraic structures in this strict sense, but fields are not. It was Lawvere (1963) who first introduced a functorial viewpoint into universal algebra. For functorial treatments of universal algebra, the reader is referred to Borceux (1994, Vol. 2, Chapter 3), Pareigis (1970, Chapter 3), or Schubert (1972, §18).

A much wider class of mathematical structures has been studied by model theorists. In particular, the model theory of finitary first-order logic is flourishing, for which the reader is referred to Chang and Keisler (1973) or Hodges (1993). For the model theory of infinitary first-order logic, the reader is referred to Dickmann (1975). The close relationship between (infinitary, many-sorted) first-order logic and sketches of Ehressmann (1968) and

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his school, as is discussed by Makkai and Paré (1989, Chapter 3), enables us to treat first-order structures functorially. In particular, limit sketches are of their logical counterpart, namely, limit theories. The duality between limit sketches and limit theories grows into the trinity of limit sketches, limit theories, and locally presentable categories, for which the reader is referred to Makkai and Paré (1989) and Adámek and Rosicky (1994). The theory of locally presentable categories was initiated by Gabriel and Ulmer (1971). The category of partially ordered sets and order-preserving mappings and that of Banach spaces and contractive linear transformations are locally presentable categories, while the category of fields and homomorphisms is not.

It is highly interesting to note that Grothendieck's sites, which are the central concept in his functorial approach to algebraic geometry (Artin *et al.*, 1972), are limit sketches, so that Grothendieck toposes, consisting of models of sites as limit sketches, are locally presentable categories. In a previous paper (Nishimura, 1996) we showed how to quantize Grothendieck toposes logically. The principal concern of this paper is to show that the method can be generalized to limit sketches without much difficulty. At present we do not know how to generalize the method to sketches in general, nor are we sure whether general sketches admit of logical quantization at all.

The organization of this paper goes as follows: Section 1 is devoted to a review of infinitary logic, sketches, accessible categories (a generalization of locally presentable categories), and their trinity. After limit sketches are Booleanized in Section 2, the relationship between two Booleanizations of limit sketches with respect to possibly distinct complete Boolean algebras is discussed in Section 3. The last section is devoted to quantizing limit sketches logically.

We close this introduction by reviewing some prerequisites and fixing notation and terminology.

0.1. Set Theory

Unless stated to the contrary, we will work within the Zermelo–Fraenkel set theory with the axiom of choice, for which the reader is referred to a standard textbook on set theory such as Jech (1978). The term "set" should be strictly distinguished from the term "class." An *ordinal* is regarded as the special set consisting exactly of all its preceding ordinals. Ordinals are denoted by α, β, γ, δ, ... The first infinite ordinal is denoted by ω. A *cardinal* is put down as a special kind of ordinal. The domain of a function f is denoted by dom(f). Such a notation as $\{x_\beta\}_{\beta\in\alpha}$ is usually to be regarded as a function whose domain is α and which assigns x_β to each $\beta \in \alpha$. However it is sometimes put down as the set whose elements are all x_β's ($\beta \in \alpha$). An infinite cardinal λ is called *regular* if for any $\alpha \in \lambda$ and any family $\{\gamma_\beta\}_{\beta \in \alpha}$ with $\gamma_\beta \in \lambda$ for all $\beta \in \alpha$, we have $\bigcup_{\beta \in \alpha} \gamma_\beta \in \lambda$.

We assume also that there exists a set **V** closed under every fundamental set-theoretic operations. Such a set is called a *universe*, and its usage to dodge the famous paradoxes of set theory is a common practice in category theory. For the exact definition of a universe, the reader is referred to MacLane (1971, Chapter I, §6), Schubert (1972, §3.2), or Borceux (1994, Vol. 1, §1.1). Sets belonging to **V** are called *small*. The adjective "small" is applied to structures whose underlying sets are small. The category of small sets and small functions is denoted by **Ens**.

0.2. Boolean Locales

The category of small complete Boolean algebras and their complete Boolean homomorphisms is denoted by **Bool**. Its dual category is denoted by **BLoc**. The objects of **BLoc** are called *Boolean locales* and are denoted by X, Y, ... The morphisms of **BLoc** are denoted by f, g, ... If a Boolean locale X is to be put down as an object of **Bool**, it is denoted by $\mathscr{P}(X)$ for emphasis, though X and $\mathscr{P}(X)$ denote the same entity. The morphism of **Bool** corresponding to a morphism f: X $\to$ Y of **BLoc** is denoted by $\mathscr{P}^*(f)$, while the right-adjoint of $\mathscr{P}^*(f)$: $\mathscr{P}(Y) \to \mathscr{P}(X)$, whose existence is guaranteed by Theorem 2.1 of Nishimura (1993b), is denoted by $\mathscr{P}_*(f)$. A *manual of Boolean locales* is a small subcategory of **BLoc** satisfying certain mild constraints, as was the case in a previous paper (Nishimura, 1995c).

0.3. X-$\underline{\text{Sets}}$ and X-Sets

Let X be a Boolean locale, which shall be fixed throughout this subsection. We will often write **B** for $\mathscr{P}(X)$. An X-$\underline{\text{set}}$ is a pair $(U, [\![\cdot = \cdot]\!]_X^U)$ of a set U and a function $[\![\cdot = \cdot]\!]_X^U$: $U \times U \to \mathbf{B}$ abiding by the following conditions:

(0.3.1) $[\![x = y]\!]_X^U = [\![y = x]\!]_X^U$
(0.3.2) $[\![x = y]\!]_X^U \wedge [\![y = z]\!]_X^U \leq [\![x = z]\!]_X^U$

for all $x, y, z \in U$. We will often write $[\![x = y]\!]_X$, $[\![x = y]\!]^U$, or simply $[\![x = y]\!]$ for $[\![x = y]\!]_X^U$, unless confusion may arise. We will often write $E_X^U x$, $E^U x$, $E_X x$, or Ex for $[\![x = x]\!]$. An X-$\underline{\text{set}}$ $(U, [\![\cdot = \cdot]\!])$ is often represented simply by its underlying set U. Given X-$\underline{\text{sets}}$ $(U, [\![\cdot = \cdot]\!]^U)$ and $(V, [\![\cdot = \cdot]\!]^V)$, we write $(U, [\![\cdot = \cdot]\!]^U) \times_X (V, [\![\cdot = \cdot]\!]^V)$ for the X-set $(U \times_X V, [\![\cdot = \cdot]\!]^{U \times_X V})$, where:

(0.3.3) $U \times_X V = \{(x, y) \in U \times V \mid E^U x = E^V y\}$.
(0.3.4) $[\![(x, y) = (x', y')]\!]^{U \times_X V} = [\![x = x']\!]^U \wedge [\![y = y']\!]^V$ for all (x, y), $(x', y') \in U \times_X V$.

To make the set of all small X-$\underline{\text{sets}}$ a category $\underline{\mathbf{BEns}}_i(\mathrm{X})$ ($i = 0, 1$), we need to define a morphism from a small X-$\underline{\text{set}}$ U to a small X-$\underline{\text{set}}$ V, which is to be a function $\delta: U \times V \to \mathbf{B}$ abiding by the following conditions:

(0.3.5) $[\![x = x']\!]^U \wedge \delta(x, y) \leq \delta(x', y)$
(0.3.6) $\delta(x, y) \wedge [\![y = y']\!]^V \leq \delta(x, y')$
(0.3.7) $\delta(x, y) \wedge \delta(x, y') \leq [\![y = y']\!]^V$
(0.3.8) $\vee_{y \in V}\, \delta(x, y) = \mathbf{E}x$

for all $x, x' \in U$ and all $y, y' \in V$.

Given an X-$\underline{\text{set}}$ $(U, [\![\cdot = \cdot]\!])$, a function $\alpha: U \to \mathbf{B}$ is called a *singleton* if it satisfies the following conditions:

(0.3.9) $\alpha(x) \wedge [\![x = y]\!] \leq \alpha(y)$
(0.3.10) $\alpha(x) \wedge \alpha(y) \leq [\![x = y]\!]$

for all $x, y \in U$. It is easy to see that each $x \in U$ gives rise to a singleton $\{x\}$ assigning to each $y \in U$ $[\![x = y]\!] \in \mathbf{B}$. The X-$\underline{\text{set}}$ $(U, [\![\cdot = \cdot]\!])$ is called an X-*set* if every singleton is of the form $\{x\}$ for a unique $x \in U$. We denote by **BEns**(X) the full subcategory of $\underline{\mathbf{BEns}}(\mathrm{X})$ whose objects are all X-sets. As is discussed in Goldblatt (1979, §11.9 and §14.7), the categories $\underline{\mathbf{BEns}}(\mathrm{X})$ and **BEns**(X) are toposes. As we have discussed in Nishimura (1995b, Theorem 1.2), there is a geometric morphism $(\mathbf{i}_{\mathbf{BEns}}[\mathrm{X}], \mathbf{a}_{\mathbf{BEns}}[\mathrm{X}])$ from **BEns**(X) to $\underline{\mathbf{BEns}}(\mathrm{X})$.

Let U be a small X-$\underline{\text{set}}$ and V a small X-set. Then there is a natural bijection between the morphisms from U to V in **BEns**(X) and the functions $f: U \to V$ yielding the following conditions:

(0.3.11) $[\![x = y]\!]^U \leq [\![f(x) = f(y)]\!]^V$
(0.3.12) $E^V f(x) \leq E^U x$

for all $x, y \in U$. The reader is referred to Goldblatt (1979, §14.7) for the detailed construction of this well-known bijection.

Let f: $\mathrm{X}_- \to \mathrm{X}_+$ be a morphism in **BLoc**. Then the assignment

$$(U, [\![\cdot = \cdot]\!]^U) \in \mathrm{Ob}\, \underline{\mathbf{BEns}}(\mathrm{X}_+) \mapsto (U, \mathscr{P}^*(\mathrm{f})([\![\cdot = \cdot]\!]^U)) \in \mathrm{Ob}\, \underline{\mathbf{BEns}}(\mathrm{X}_-)$$

naturally induces a functor $\underline{\underline{\mathrm{f}}}^*: \underline{\mathbf{BEns}}(\mathrm{X}_+) \to \underline{\mathbf{BEns}}(\mathrm{X}_-)$, which in turns gives rise to functors

$$\underline{\mathrm{f}}^* = \underline{\underline{\mathrm{f}}}^* \circ \mathbf{i}_{\mathbf{BEns}}[\mathrm{X}_+]: \mathbf{BEns}(\mathrm{X}_+) \to \underline{\mathbf{BEns}}(\mathrm{X}_-)$$

$$\mathrm{f}^* = \mathbf{a}_{\mathbf{BEns}}[\mathrm{X}_-] \circ \underline{\mathrm{f}}^*: \mathbf{BEns}(\mathrm{X}_+) \to \mathbf{BEns}(\mathrm{X}_-)$$

On the other hand, the assignment

$$(V, [\![\cdot = \cdot]\!]^V) \in \mathrm{Ob}\, \mathbf{BEns}(\mathrm{X}_-)$$
$$\mapsto \mathbf{a}_{\mathbf{BEns}}[\mathrm{X}_+](V, \mathscr{P}_*(\mathrm{f})([\![\cdot = \cdot]\!]^V)) \in \mathrm{Ob}\, \mathbf{BEns}(\mathrm{X}_+)$$

naturally induces a functor f_*: **BEns**(X_-) → **BEns**(X_+). As we have discussed in Nishimura (1993b, §2), the pair (f_*, f^*) forms a geometric morphism from **BEns**(X_-) to **Bens**(X_+), i.e., $f^* \dashv f_*$ and f^* is left-exact. Since the geometric morphism (f_*, f^*): **BEns**(X_-) → **BEns**(X_+) corresponds to the morphism f: X_- → X_+ in **BLoc** under Theorem 2.6 of Nishimura (1993b) and f is open by Theorem 2.13 of Nishimura (1993b), the geometric morphism (f_*, f^*) is essential due to Exercise 2.13.8 of Borceux (1994, Vol. 3) in the sense that f^* has a left-adjoint $f_!$: **BEns**(X_-) → **BEns**(X_+). In particular, the functor f^*: **BEns**(X_+) → **BEns**(X_-) preserves not only arbitrary colimits, but also arbitrary limits by dint of Theorem 1 of MacLane (1971, Chapter V, §5).

0.4. Two Transfer Principles

Let X be a Boolean locale with **B** = $\mathscr{P}$(X). As we have discussed in Nishimura (1993b), the topos **BEns**(X) is equivalent to the category of sets and functions within the Scott–Solovay universe $\mathbf{V}^{(\mathbf{B})}$. As Jech (1978, Theorem 43) and others have discussed, the universe $\mathbf{V}^{(\mathbf{B})}$ enjoys ZFC (Zermelo–Fraenkel set theory with the axiom of choice), which is the core principle of Boolean mathematics. For flourishing Boolean mathematics, the reader is referred, e.g., to Nishimura (1984, 1991, 1992, 1993a), Ozawa (1983, 1984, 1985), Smith (1984), and the Bible of Boolean mathematics, namely, Takeuti (1978). Since every branch of mathematics, ranging from algebraic geometry to functional analysis, is in principle to be developed within ZFC, the Scott–Solovay universe $\mathbf{V}^{(\mathbf{B})}$ and therefore its equivalent **BEns**(X) enjoy all classical mathematics (=mathematics based on classical logic). This transfer principle from standard mathematics to Boolean mathematics is designated the *Zermelo–Fraenkel transfer principle* or ZFTP for short. The application of the transfer principle is usually denominated *Booleanization.*

Let f: X_- → X_+ be a morphism of **BLoc**. Due to Theorem 2.13 of Nishimura (1993b), f is open, so that the geometric morphism (f_*, f^*): **BEns**(X_-) → **BEns**(X_+) is also open by Proposition 2 of MacLane and Moerdijk (1992, Chapter IX, §7). This implies that every finitary first-order property holding in a (many-sorted) first-order structure $\mathscr{A}$ in **BEns**(X_+) persists in the derived first-order structure $f^*\mathscr{A}$ in **BEns**(X_-), as is claimed in Corollary 4 of MacLane and Moerdijk (1992, Chapter X, §3). This transfer principle is designated the *first-order transfer principle* or FOTP for short.

0.5. X-Categories

Let X be a Boolean locale. The interpretation of the notion of a category within the topos **BEns**(X) gives rise to that of a small X-category, as discussed in Nishimura (1995c, §1). By way of example, the totality of **BEns**(X_p)'s [$p \in \mathscr{P}(X)$] lumps together to form an X-category $\mathscr{BE}ns$(X), as dealt with in

Nishimura (1995c, Example 1.1). The reader is referred to Nishimura (1995c) for the details of the theory of X-categories. We use $\cong_X$ for the natural X-isomorphism between X-functors. Given X-categories $\mathscr{A}$ and $\mathscr{B}$, a *partial X-functor* from $\mathscr{A}$ to $\mathscr{B}$ is an X_p-functor from $\mathscr{A}\lceil p$ to $\mathscr{B}\lceil p$ for some $p \in \mathscr{P}(X)$. The totality of partial X-functors from a small X-category $\mathscr{A}$ to $\mathscr{BEns}(X)$ naturally forms an X-category to be denoted by $\mathscr{BPreSh}(\mathscr{A})$. The canonical contravariant Yoneda embedding of $\mathscr{A}$ into $\mathscr{BPreSh}(\mathscr{A})$ is denoted by **y**. The Booleanizations of left adjoint and right adjoint functors, discussed in Nishimura (n.d.-a), are called left and right X-adjoints respectively.

Let $\mathfrak{f}: X_- \to X_+$ be a morphism of Boolean locales. The notion of an X-functor was generalized in Nishimura (1995c, §2) to that of an $\mathfrak{f}$-functor from an X_+-category $\mathscr{C}_+$ to an X_--category $\mathscr{C}_-$. By way of example, the functors $\mathfrak{f}_p^*$: $\mathbf{BEns}((X_+)_p) \to \mathbf{BEns}((X_-)_{\mathscr{P}(\mathfrak{f})(p)})$ for all $p \in \mathscr{P}(X_+)$ lump together to form an $\mathfrak{f}$-functor $\mathfrak{f}^*_{\mathscr{BEns}}$: $\mathscr{BEns}(X_+) \to \mathscr{BEns}(X_-)$, where $\mathfrak{f}_p$ denotes the morphism of Boolean locales from $(X_-)_{\mathscr{P}(\mathfrak{f})(p)}$ to $(X_+)_p$ naturally induced by $\mathfrak{f}$. The $\mathfrak{f}$-functor $\mathfrak{f}^*_{\mathscr{BEns}}$ naturally induces such $\mathfrak{f}$-functors as $\mathfrak{f}^*_{\mathscr{BRng}}$, which was discussed amply Nishimura (1995c). Unless confusion may occur, the superscripts in such notations as $\mathfrak{f}^*_{\mathscr{BEns}}$ and $\mathfrak{f}^*_{\mathscr{BRng}}$ are often omitted, so that the notation $\mathfrak{f}^*$ enjoys a bit of polysemy. We use $\cong_{\mathfrak{f}}$ for the natural $\mathfrak{f}$-isomorphism between X-functors.

1. FIRST-ORDER STRUCTURES

In this section we work within the universe **V**. This implies that a set means a small set unless stated to the contrary. λ denotes a regular cardinal in this universe.

This section is essentially a review. For infinitary logic the reader is referred to Dickmann (1975). For sketches and accessible categories the reader is referred to Adámek and Rosicky (1994), Borceux (1994, Chapter 6 of Vol. 1 and Chapter 5 of Vol. 2 in particular), and Makkai and Paré (1989).

1.1. Infinitary Logic

An (infinitary many-sorted) formal language L is determined by three disjoint sets, namely, a set L_{sor} of *sorts*, a set L_{rel} of *relation symbols*, and a set L_{ope} of *operation symbols*. Every relation symbol R is assigned its *arity* ari(R), which is a function from an ordinal α to L_{sor}. Similarly, every operation symbol o is assigned its *arity* ari(o) and its *value sort* v-sor(o). The former is a function from an ordinal α to L_{sor}, while the latter is an element of L_{sor}.

We assume that an abundant supply of variables of each sort s is chosen and fixed. The notions of a *term* τ and its *value sort* v-sor(τ) are defined simultaneously by induction as follows:

(1.1.1) Each variable x of sort s is a term of value sort s.
(1.1.2) If $\mathscr{o}$ is an operation symbol of arity ξ and value sort s and if σ_α is a term of value sort $\xi(\alpha)$ for each $\alpha \in \mathrm{dom}(\xi)$, then the pair $\langle \mathscr{o}, \{\sigma_\alpha\}_{\alpha \in \mathrm{dom}(\xi)} \rangle$ is a term of value sort s.

The notion of an *atomic formula* is defined as follows:

(1.1.3) If R is a relation symbol of arity ξ and τ_α is a term of value sort $\xi(\alpha)$ for each $\alpha \in \mathrm{dom}(\xi)$, then the pair $\langle R, \{\tau_\alpha\}_{\alpha \in \mathrm{dom}(\xi)} \rangle$ is an atomic formula.
(1.1.4) If σ and τ are terms of the same value sort, then the triple $\langle =, \sigma, \tau \rangle$ is an atomic formula.

The atomic formula $\langle =, \sigma, \tau \rangle$ in (1.1.4) is often abbreviated to $\sigma = \tau$. The class of formulas is constructed from atomic formulas by using logical operators $\neg$ (negation), $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction), $\forall$ (universal quantifier), and $\exists$ (existential quantifier). Exactly speaking, the notion of a *formula* φ is defined inductively as follows:

(1.1.5) An atomic formula is a formula.
(1.1.6) If φ is a formula, then the pair $\langle \neg, \varphi \rangle$ is also a formula.
(1.1.7) If φ and ψ are formulas, then the triple $\langle \rightarrow, \varphi, \psi \rangle$ is also a formula.
(1.1.8) If α is an ordinal and φ_β is a formula for each $\beta \in \alpha$, then the pair $\langle \wedge, \{\varphi_\beta\}_{\beta \in \alpha} \rangle$ is a formula.
(1.1.9) If α is an ordinal and φ_β is a formula for each $\beta \in \alpha$, then the pair $\langle \vee, \{\varphi_\beta\}_{\beta \in \alpha} \rangle$ is a formula.
(1.1.10) If φ is a formula, α is an ordinal, and x_β is a variable for each $\beta \in \alpha$, then the triple $\langle \forall, \{x_\beta\}_{\beta \in \alpha}, \varphi \rangle$ is a formula.
(1.1.11) If φ is a formula, α is an ordinal, and x_β is a variable for each $\beta \in \alpha$, then the triple $\langle \exists, \{x_\beta\}_{\beta \in \alpha}, \varphi \rangle$ is a formula.

We will often write $\neg\varphi$ and $\varphi \rightarrow \psi$ for the formulas $\langle \neg, \varphi \rangle$ in (1.1.6) and $\langle \rightarrow, \varphi, \psi \rangle$ in (1.1.7), respectively. The formulas $\langle \wedge, \{\varphi_\beta\}_{\beta \in \alpha} \rangle$ in (1.1.8) and $\langle \vee, \{\varphi_\beta\}_{\beta \in \alpha} \rangle$ in (1.1.9) are often abbreviated to $\wedge_{\beta \in \alpha}\varphi_\beta$ and $\vee_{\beta \in \alpha}\varphi_\beta$, respectively, while the formulas $\langle \forall, \{x_\beta\}_{\beta \in \alpha}, \varphi \rangle$ in (1.1.10) and $\langle \exists, \{x_\beta\}_{\beta \in \alpha}, \varphi \rangle$ in (1.1.11) are often designated $(\forall_{\beta \in \alpha} x_\beta)\varphi$ and $(\exists_{\beta \in \alpha} x_\beta)\varphi$, respectively. The notation $(\exists!_{\beta \in \alpha} x_\beta)\varphi$ is an abbreviation of

$$(\exists_{\beta \in \alpha} x_\beta)\varphi \wedge (\varphi \wedge \varphi(\{y_\beta / x_\beta\}_{\beta \in \alpha}) \rightarrow \wedge_{\beta \in \alpha} x_\beta = y_\beta)$$

where y_β is a variable of the same sort as x_β not occurring in φ for each $\beta \in \alpha$ and $\varphi(\{y_\beta / x_\beta\}_{\beta \in \alpha})$ denotes the formula obtained from φ by replacing every free occurrence of x_β by y_β.

Now we define the set $\mathrm{Var}(\tau)$ of free variables in a term τ by the construction of τ as follows:

(1.1.12) $\mathrm{Var}(x) = \{x\}$ for each variable x.

(1.1.13) If $\mathscr{o}$ is an operation symbol of arity ξ and σ_α is a term of value sort $\xi(\alpha)$ for each $\alpha \in \mathrm{dom}(\xi)$, then

$$\mathrm{Var}(\langle \mathscr{o}, \{\tau_\alpha\}_{\alpha \in \mathrm{dom}(\xi)}\rangle) = \bigcup\{\mathrm{Var}(\tau_\alpha) \mid \alpha \in \mathrm{dom}(\xi)\}$$

Now we define the set $\mathrm{Var}(\varphi)$ of free variables in a formula φ by the construction of φ as follows:

(1.1.14) If R is a relation symbol of arity ξ and τ_α is a term of value sort $\xi(\alpha)$ for each $\alpha \in \mathrm{dom}(\xi)$, then

$$\mathrm{Var}(\langle R, \{\tau_\alpha\}_{\alpha \in \mathrm{dom}(\xi)}\rangle) = \bigcup\{\mathrm{Var}(\tau_\alpha) \mid \alpha \in \mathrm{dom}(\xi)\}$$

(1.1.15) If σ and τ are terms of the same value sort, then

$$\mathrm{Var}(\langle =, \sigma, \tau\rangle) = \mathrm{Var}(\sigma) \cup \mathrm{Var}(\tau)$$

(1.1.16) If φ is a formula, then

$$\mathrm{Var}(\langle \neg, \varphi\rangle) = \mathrm{Var}(\varphi)$$

(1.1.17) If φ and ψ are formulas, then

$$\mathrm{Var}(\langle \rightarrow, \varphi, \psi\rangle) = \mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi)$$

(1.1.18) If α is an ordinal and φ_β is a formula for each $\beta \in \alpha$, then

$$\mathrm{Var}(\langle \wedge, \{\varphi_\beta\}_{\beta \in \alpha}\rangle) = \bigcup\{\mathrm{Var}(\varphi_\beta) \mid \beta \in \alpha\}$$

(1.1.19) If α is an ordinal and φ_β is a formula for each $\beta \in \alpha$, then

$$\mathrm{Var}(\langle \vee, \{\varphi_\beta\}_{\beta \in \alpha}\rangle) = \bigcup\{\mathrm{Var}(\varphi_\beta) \mid \beta \in \alpha\}$$

(1.1.20) If φ is a formula, α is an ordinal, and x_β is a variable for each $\beta \in \alpha$, then

$$\mathrm{Var}(\langle \forall, \{x_\beta\}_{\beta \in \alpha}, \varphi\rangle) = \mathrm{Var}(\varphi) - \{x_\beta\}_{\beta \in \alpha}$$

(1.1.21) If φ is a formula, α is an ordinal, and x_β is a variable for each $\beta \in \alpha$, then

$$\mathrm{Var}(\langle \exists, \{x_\beta\}_{\beta \in \alpha}, \varphi\rangle) = \mathrm{Var}(\varphi) - \{x_\beta\}_{\beta \in \alpha}$$

A formula φ with $\mathrm{Var}(\varphi) = \phi$ is called a *sentence*. A set of sentences is called a *theory*.

A *structure A* for a given formal language L or simply an *L-structure* consists of the following three entities:

(1.1.22) An assignment to each sort s of a set A_s.
(1.1.23) An assignment to each relation symbol R with arity ξ of a subset R_A of $\Pi_{\alpha \in \mathrm{dom}(\xi)} A_{\xi(\alpha)}$.
(1.1.24) An assignment to each operation symbol o with arity ξ and value sort s of a function o_A from $\Pi_{\alpha \in \mathrm{dom}(\xi)} A_{\xi(\alpha)}$ to A_s.

Given an L-structure A, an *individual assignment* is a function I from a set of variables such that whenever x is a variable of sort s and happens to be in dom(I), then $I(x) \in A_s$. The individual assignment I can be extended to all the terms τ with $\mathrm{Var}(\tau) \subseteq \mathrm{dom}(I)$ on the construction of τ as follows:

(1.1.25) If o is an operation symbol of arity ξ and value sort s, σ_α is a term of value sort $\xi(\alpha)$ for each $\alpha \in \mathrm{dom}(\xi)$, and τ is of the form $\langle o, \{\sigma_\alpha\}_{\alpha \in \mathrm{dom}(\xi)}\rangle$, then $I(\tau) = o_A(\{I(\sigma_\alpha)\})$.

The basic Tarskian semantical notion of $A \models \varphi[I]$ with $\mathrm{Var}(\varphi) \subseteq \mathrm{dom}(I)$, which should read "the individual assignment I satisfies the formula φ in the structure A," can be defined on the construction of φ as follows:

(1.1.26) $A \models \langle R, \{\tau_\alpha\}_{\alpha \in \mathrm{dom}(\xi)}\rangle[I]$ iff $\{I(\tau_\alpha)\}_{\alpha \in \mathrm{dom}(\xi)} \in R_A$, where ξ is the arity of R.
(1.1.27) $A \models \langle =, \sigma, \tau\rangle[I]$ iff $I(\sigma) = I(\tau)$.
(1.1.28) $A \models \langle \neg, \varphi\rangle[I]$ iff it is not the case that $A \models \varphi[I]$.
(1.1.29) $A \models \langle \rightarrow, \varphi, \psi\rangle[I]$ iff $A \models \langle \neg, \varphi\rangle[I]$ or $A \models \psi[I]$.
(1.1.30) $A \models \langle \wedge, \{\varphi_\beta\}_{\beta \in \alpha}\rangle[I]$ iff $A \models \varphi_\beta[I]$ for all $\beta \in \alpha$.
(1.1.31) $A \models \langle \vee, \{\varphi_\beta\}_{\beta \in \alpha}\rangle[I]$ iff $A \models \varphi_\beta[I]$ for some $\beta \in \alpha$.
(1.1.32) $A \models \langle \forall, \{x_\beta\}_{\beta \in \alpha}, \varphi\rangle[I]$ iff $A \models \varphi[I']$ for all extensions I' of I with $\{x_\beta\}_{\beta \in \alpha} \subseteq \mathrm{dom}(I')$.
(1.1.33) $A \models \langle \exists, \{x_\beta\}_{\beta \in \alpha}, \varphi\rangle[I]$ iff $A \models \varphi[I']$ for some extension I' of I with $\{x_\beta\}_{\beta \in \alpha} \subseteq \mathrm{dom}(I')$.

We note that if I and I' are individual assignments such that dom(I) and dom(I') contain Var(φ), and I and I' agree on Var(φ), then $A \models \varphi[I]$ iff $A \models \varphi[I']$. In particular, if φ is a sentence, whether $A \models \varphi[I]$ or not is independent of I, so that we can safely define the semantical notion of $A \models \varphi$, which should read "the sentence φ is true in A" or "A is a model of φ," to be $A \models \varphi[I]$ for some and therefore for all individual assignments I. If T is a theory and $A \models \varphi$ for all $\varphi \in T$, then A is called a *model* of T.

Given two L-structures A and B, a *homomorphism* f from A to B is a family $\{f_s\}_{s \in L_{\mathrm{sor}}}$ of functions $f_s\colon A_s \rightarrow B_s$ yielding the following conditions:

(1.1.34) For each operation symbol o with arity ξ and value sort s, we have

$$o_B(\{f_{\xi(\alpha)}(x_\alpha)\}_{\alpha \in \mathrm{dom}(\xi)}) = f_s(o_A(\{x_\alpha\}_{\alpha \in \mathrm{dom}(\xi)}))$$
$$\text{for all} \quad \{x_\alpha\}_{\alpha \in \mathrm{dom}(\xi)} \in \Pi_{\alpha \in \mathrm{dom}(\xi)} A_{\xi(\alpha)}$$

(1.1.35) For each relation symbol R with arity ξ, we have

$$\{f_{\xi(\alpha)}(x_\alpha)\}_{\alpha \in \mathrm{dom}(\xi)} \in R_B \qquad \text{for all} \quad \{x_\alpha\}_{\alpha \in \mathrm{dom}(\xi)} \in R_A$$

We denote by **Str** L the category of all L-structures and homomorphisms. Given a theory T, its full subcategory of models of T is denoted by **Mod** T.

A formula is called *positive-existential* if it is built up from atomic formulas with (repeated) use of the operators $\wedge$, $\vee$, and $\exists$. A sentence is called a *basic sentence* if it is of the form $(\forall_{\beta \in \alpha} x_\beta)(\varphi \rightarrow \psi)$ with positive-existential formulas φ and ψ. A sentence is called a *limit sentence* if it is of the form $(\forall_{\beta \in \alpha} x_\beta)(\varphi \rightarrow (\exists!_{\delta \in \gamma} x_\delta)\psi)$ with φ and ψ being conjunctions of atomic formulas. A sentence is called *equational* if it is of the form $(\forall_{\beta \in \alpha} x_\beta)(\sigma = \tau)$ with terms σ and τ. A theory is called *basic* (*limit*, *equational*, resp.) if it consists only of basic (limit, equational, resp.) sentences. As was claimed by Makkai and Paré (1989, Proposition 3.2.8), we do not lose generality considerably even if we restrict consideration to basic theories.

Up to now we have discussed what is called the ∞-*logic*. We conclude this subsection by discussing how to modify the above discussion so as to get what is called the λ-*logic*. In particular, if $\lambda = \omega$, then we will get the *f initary logic*.

A formal language L is called a λ-*formal language* if $\mathrm{dom}(\mathrm{ari}(R)) < \lambda$ for any $R \in L_{\mathrm{rel}}$ and $\mathrm{dom}(\mathrm{ari}\ o)) < \lambda$ for any $o \in L_{\mathrm{ope}}$. Now we let L be a λ-formal language. By restricting $\alpha < \lambda$ in (1.1.8)–(1.1.10) in the above inductive definition of formulas, we get the notion of a λ-*formula*. A λ-formula φ with $\mathrm{Var}(\varphi) = \phi$ is called a λ-*sentence*. A theory T over a λ-formal language L is called a λ-*theory* if it consists only of λ-sentences.

1.2. Sketch

The notion of a sketch was introduced by Ehresmann (1968) and its theory has been developed by his French school. Formally speaking, a *sketch* is a triple (**S**, **L**, **C**) of a small category **S**, a set **L** of cones in **S**, and a set **C** of cocones in **S**. Given two sketches (**S**, **L**, **C**) and (**S**′, **L**′, **C**′), a *sketch map* from (**S**, **L**, **C**) to (**S**′, **L**′, **C**′) is a functor F: **S** $\rightarrow$ **S**′ mapping cones in L into L' and cocones in **C** into **C**′. The category of sketches and sketch maps is denoted by **Sketch**. A *model* of a sketch (**S**, **L**, **C**) is a functor from **S** to the category of sets and functions mapping cones in **L** into limiting cones and cocones in **C** into colimiting cocones. The category of models of a sketch (**S**, **L**, **C**) and natural transformations is denoted by **Mod(S, L, C)**.

A sketch (**S**, **L**, **C**) is called a *limit sketch* if **C** $= \phi$, in which it is natural to denote it simply by (**S**, **L**). A limit sketch (**S**, **L**) is called a λ-*limit sketch* if the size of each cone in **L** is less than λ. A limit sketch (**S**, **L**) is called a *finite-product sketch* if every cone in **L** is a cone over a finite discrete diagram.

The relationship between sketches and the ∞-logic is elementary and fundamental.

Theorem 1.2.1. For any sketch (**S**, **L**, **C**), there exists a formal language L and a basic theory T over L such that the category **Mod**(**S**, **L**, **C**) is equivalent to the category **Mod** T. Conversely, for any basic theory T over a formal language L, there exists a sketch (**S**, **L**, **C**) such that the category **Mod** T is equivalent to the category **Mod**(**S**, **L**, **C**).

By way of example, since the theory of fields is basic [cf. Adámek and Rosicky (1994), Example 5.32.(5)], the category of fields and homomorphisms is equivalent to **Mod**(**S**, **L**, **C**) for some sketch (**S**, **L**, **C**). The details of such a sketch (**S**, **L**, **C**) are given in Barr and Wells (1990, §7.9). For the proof of the above theorem, the reader is referred to Makkai and Paré (1989, Theorem 3.2.1).

The proof of Theorem 1.2.1 can be modified readily to yield some other duality theorems between well-behaved classes of sketches and corresponding sublogics of the ∞-logic. In particular, we have the following:

Theorem 1.2.2. A category **A** is equivalent to **Mod**(**S**, **L**) for some λ-limit sketch (**S**, **L**) iff it is equivalent to **Mod** T for some limit λ-theory T.

By way of example, since the theory of partially ordered sets is an ω-limit theory, the category of partially ordered sets and order-preserving functions is equivalent to **Mod**(**S**, **L**) for some ω-limit sketch (**S**, **L**). For the details of such (**S**, **L**), the reader is referred to Adámek and Rosicky (1994), Example 1.50.(5). Another interesting example of ω-limit sketch is what is called a site of Grothendieck, whose models as well as natural transformations among them constitute its Grothendieck topos.

Theorem 1.2.3. A category **A** is equivalent to **Mod**(**S**, **L**) for some finite-product sketch (**S**, **L**) iff it is equivalent to **Mod** T for some equational ω-theory T.

By way of example, the finite-product sketch for the theory of semigroups can be seen in Barr and Wells (1990, §7.2).

The last theorem was the motif of Lawvere's (1963) dissertation, which first dealt with functorial semantics of algebraic theories, shedding new light upon what is called universal algebra and paving the way to the sketches already discussed and to accessible categories, which are discussed next.

1.3. Accessible Categories

The duality between sketches and the ∞-logic can be extended to the trinity among the above two and accessible categories. The notion of an accessible category was introduced by Lair (1981) under the name "catégorie modelable" so as to characterize sketches in the spirit of Gabriel and Ulmer (1971).

An object A of a category A is called *λ-presentable* if the representable functor $\mathbf{A}(A, ?)$: $\mathbf{A} \to \mathbf{Ens}$ preserves λ-filtered colimits existing in $\mathbf{A}$. A category $\mathbf{A}$ is called *λ-accessible* if it is subject to the following conditions:

(1.3.1) $\mathbf{A}$ has λ-filtered colimits.
(1.3.2) There exists a small full subcategory $\mathbf{B}$ of $\mathbf{A}$ consisting of λ-representable objects such that every object of $\mathbf{A}$ is a λ-filtered colimit of a diagram of objects in $\mathbf{B}$.

A category $\mathbf{A}$ is called *accessible* if it is λ-accessible for some regular cardinal λ. As is well known, every poset P can be regarded as a category in which there exists at most one arrow from p to q for any ordered pair (p, q) of elements of P. In this light ω-accessible posets are exactly Scott domains, for which the reader is referred to Adámek and Rosicky (1994), Example 2.3.(2). Another interesting example of accessible category is the category **Hilb** of Hilbert spaces and contractions, for which the reader is referred to Makkai and Paré (1989, Proposition 3.4.2).

Theorem 1.3.1. A category is accessible iff it is equivalent to $\mathbf{Mod}(\mathbf{S}, \mathbf{L}, \mathbf{C})$ for some sketch $(\mathbf{S}, \mathbf{L}, \mathbf{C})$.

For the proof of the above theorem the reader is referred to Makkai and Paré (1989, Theorems 3.3.4 and 4.3.2).

A λ-accessible category is called *locally λ-presentable* if it is cocomplete. A category is called *locally presentable* if it is locally λ-presentable for some regular cardinal λ. In the above light of posets as categories, locally ω-presentable posets are exactly algebraic lattices. We note in passing that a locally λ-presentable category can be defined as a complete λ-accessible category, for which the reader is referred to Borceux (1994, Vol. 2, Theorem 5.5.8).

Theorem 1.3.2. A category is locally λ-presentable iff it is equivalent to $\mathbf{Mod}(\mathbf{S}, \mathbf{L})$ for some λ-limit sketch $(\mathbf{S}, \mathbf{L})$.

For the proof of the above theorem the reader is referred to Adámek and Rosicky (1994, Theorem 5.30).

1.4. Limit Sketches

Since we are concerned with logical quantizations of limit sketches, it is natural to conclude this section with a brief treatment of limit sketches.

Let (S, L) be a limit sketch. We often denote **Mod(S, L)** by **Sh(S, L)**, while the category of functors from **S** to **Ens** and natural transformations is often denoted by **PreSh(S)**.

Theorem 1.4.1. Let (S, L) be a limit sketch. Then the inclusion functor $\mathbf{i}_{\mathbf{L}}$: **Sh(S, L)** → **PreSh(S)** has a left adjoint $\mathbf{a}_{\mathbf{L}}$: **PreSh(S)** → **Sh(S, L)** such that it is the identity functor on **Sh(S, L)** (i.e., $\mathbf{a}_{\mathbf{L}} \circ \mathbf{i}_{\mathbf{L}} = \mathbf{i}_{\mathbf{L}}$).

For the proof of the above theorem, the reader is referred to Adámek and Rosicky (1994), Example 1.33.(8) and Theorem 1.39.

The following is an example of Kan extensions.

Theorem 1.4.2. Let φ: $\mathbf{S}_+ \to \mathbf{S}_-$ be a functor of small categories. Then the induced functor φ_*: $\mathbf{PreSh}(\mathbf{S}_-) \to \mathbf{PreSh}(\mathbf{S}_+)$ has a left adjoint φ^*: $\mathbf{PreSh}(\mathbf{S}_+) \to \mathbf{PreSh}(\mathbf{S}_-)$.

For the proof of the above theorem, the reader is referred to MacLane (1971, Chapter X, §3, Theorem 1).

Theorem 1.4.3. Let φ: $(\mathbf{S}_+, \mathbf{L}_+) \to (\mathbf{S}_-, \mathbf{L}_-)$ be a sketch map. Since the functor φ_*: $\mathbf{PreSh}(\mathbf{S}_-) \to \mathbf{PreSh}(\mathbf{S}_+)$ maps $\mathbf{Sh}(\mathbf{S}_-, \mathbf{L}_-)$ into $\mathbf{Sh}(\mathbf{S}_+, \mathbf{L}_+)$, it induces a functor $\tilde{\varphi}_*$: $\mathbf{Sh}(\mathbf{S}_-, \mathbf{L}_-) \to \mathbf{Sh}(\mathbf{S}_+, \mathbf{L}_+)$ with $\tilde{\varphi}_* = \mathbf{a}_{\mathbf{L}_+} \circ \varphi_* \circ \mathbf{i}_{\mathbf{L}_-}$. The functor $\tilde{\varphi}^* = \mathbf{a}_{\mathbf{L}_-} \circ \varphi^* \circ \mathbf{i}_{\mathbf{L}_+}$ is left adjoint to $\tilde{\varphi}_*$.

Proof. For any $x \in \mathrm{Ob}\, \mathbf{Sh}(\mathbf{S}_+, \mathbf{L}_+)$ and $y \in \mathrm{Ob}\, \mathbf{Sh}(\mathbf{S}_-, \mathbf{L}_-)$, we have

$$\begin{aligned}
&\mathbf{Sh}(\mathbf{S}_-, \mathbf{L}_-)(\tilde{\varphi}^* x, y) \\
&\quad = \mathbf{Sh}(\mathbf{S}_-, \mathbf{L}_-)((\mathbf{a}_{\mathbf{L}_-} \circ \varphi^* \circ \mathbf{i}_{\mathbf{L}_+})x, y) \\
&\quad \cong \mathbf{S}_-((\varphi^* \circ i_{\mathbf{L}_+})x, \mathbf{i}_{\mathbf{L}_-} y) \qquad \text{(Theorem 1.4.1)} \\
&\quad \cong \mathbf{S}_+(\mathbf{i}_{\mathbf{L}_+} x, (\varphi_* \circ \mathbf{i}_{\mathbf{L}_-})y) \qquad \text{(Theorem 1.4.2)} \\
&\quad \cong \mathbf{Sh}(\mathbf{S}_+, \mathbf{L}_+)(x, (\mathbf{a}_{\mathbf{L}_+} \circ \varphi_* \circ \mathbf{i}_{\mathbf{L}_-})y) \qquad \text{(Theorem 1.4.1)} \\
&\quad = \mathbf{Sh}(\mathbf{S}_+, \mathbf{L}_+)(x, \tilde{\varphi}_* y)
\end{aligned}$$

Therefore $\tilde{\varphi}^* \dashv \tilde{\varphi}_*$. ■

Theorem 1.4.4. Let φ: $(\mathbf{S}_1, \mathbf{L}_1) \to (\mathbf{S}_2, \mathbf{L}_2)$ and ψ: $(\mathbf{S}_2, \mathbf{L}_2) \to (\mathbf{S}_3, \mathbf{L}_3)$ be sketch maps. Let $\chi = \psi \circ \varphi$. Then the functors $\tilde{\chi}^*$ and $\tilde{\psi}^* \circ \tilde{\varphi}^*$ are naturally isomorphic.

Proof. It is obvious that $\tilde{\chi}_* = \tilde{\varphi}_* \circ \tilde{\psi}_*$. Since $\tilde{\chi}^*$ is left adjoint to $\tilde{\chi}_*$ and $\tilde{\psi}^* \circ \tilde{\varphi}^*$ is left adjoint to $\tilde{\varphi}_* \circ \tilde{\psi}_*$ by Theorem 1.4.3, the functors $\tilde{\chi}^*$ and $\tilde{\psi}^* \circ \tilde{\varphi}^*$ should be naturally isomorphic by MacLane (1971, Chapter IV, Corollary 1 of Theorem 2).

Theorem 1.4.5. Let θ: $(\mathbf{S}_+, \mathbf{L}_+) \to (\mathbf{S}_-, \mathbf{L}_-)$ be a sketch map. Then the functors $\mathbf{a}_{\mathbf{L}_-} \circ \theta^*$ and $\mathbf{a}_{\mathbf{L}_-} \circ \theta^* \circ \mathbf{i}_{\mathbf{L}_+} \circ \mathbf{a}_{\mathbf{L}_+}$ from $\mathbf{PreSh}(\mathbf{S}_+)$ to $\mathbf{Sh}(\mathbf{S}_-, \mathbf{L}_-)$ are naturally isomorphic.

Proof. By taking θ: $(\mathbf{S}_+, \mathbf{L}_+) \to (\mathbf{S}_-, \mathbf{L}_-)$ for ψ: $(\mathbf{S}_2, \mathbf{L}_2) \to (\mathbf{S}_3, \mathbf{L}_3)$ in Theorem 1.4.4 and taking the identity functor of $\mathbf{S}_+$, regarded as a sketch map from $(\mathbf{S}_+, \phi)$ into $(\mathbf{S}_+, \mathbf{L}_+)$, for φ: $(\mathbf{S}_1, \mathbf{L}_1) \to (\mathbf{S}_2, \mathbf{L}_2)$ in Theorem 1.4.4, we get the desired result. ∎

2. BOOLEAN LIMIT SKETCHES

Let X be a Boolean locale, which shall be fixed throughout this section. An X-*limit* X-*sketch* or simply an X-*sketch* is a pair $(\mathcal{S}, \mathcal{L})$ of a small X-category $\mathcal{S}$ and an X-set of X-cones in $\mathcal{S}$. Given an X-sketch $(\mathcal{S}, \mathcal{L})$, the full X-subcategory of $\mathcal{BP}resh(\mathcal{S})$ whose objects are all partial X-functors $\mathcal{F}$ from $\mathcal{S}$ to $\mathcal{BE}ns$ mapping X-cones in $\mathcal{L}\lceil \mathbf{E}\mathcal{F}$ into X-limits in $\mathcal{BE}ns$ is denoted by $\mathcal{BSh}(\mathcal{S}, \mathcal{L})$ or $\mathcal{M}od_{\mathrm{X}}(\mathcal{S}, \mathcal{L})$. Given X-sketches $(\mathcal{S}_-, \mathcal{L}_-)$ and $(\mathcal{S}_+, \mathcal{L}_+)$, an X-*sketch* X-*map* from $(\mathcal{S}_+, \mathcal{L}_+)$ to $(\mathcal{S}_-, \mathcal{L}_-)$ is an X-functor from $\mathcal{S}_+$ to $\mathcal{S}_-$ mapping X-cones in $\mathcal{L}_+$ into X-cones in $\mathcal{L}_-$. We denote by **BSketch**(X) the category of X-sketches and X-sketch X-maps. As in Example 1.1 of Nishimura (1995c), the totality of $\mathbf{BSketch}(\mathrm{X}_p)$'s $[p \in \mathcal{P}(\mathrm{X})]$ constitutes an X-category to be denoted by $\mathcal{BS}ketch(\mathrm{X})$.

By simply Booleanizing Theorem 1.4.1, we have the following result.

Theorem 2.1. Let $(\mathcal{S}, \mathcal{L})$ be an X-sketch. Then the inclusion X-functor $\imath_{\mathcal{L}}: \mathcal{BSh}(\mathcal{S}, \mathcal{L}) \to \mathcal{BP}re\mathcal{Sh}(\mathcal{S})$ has a left X-adjoint $a_{\mathcal{L}}: \mathcal{BP}re\mathcal{Sh}(\mathcal{S}) \to \mathcal{BSh}(\mathcal{S}, \mathcal{L})$ such that it is the identity X-functor on $\mathcal{BSh}(\mathcal{S}, \mathcal{L})$ (i.e., $a_{\mathcal{L}} \circ \imath_{\mathcal{L}} = \imath_{\mathcal{L}}$).

By simply Booleanizing Theorem 1.4.2, we have the following result.

Theorem 2.2. Let $\varphi: \mathcal{S}_+ \to \mathcal{S}_-$ be a functor of small X-categories. Then the induced X-functor $\varphi_*: \mathcal{BP}re\mathcal{Sh}(\mathcal{S}_-) \to \mathcal{BP}re\mathcal{Sh}(\mathcal{S}_+)$ has a left X-adjoint $\varphi^*: \mathcal{BP}re\mathcal{Sh}(\mathcal{S}_+) \to \mathcal{BP}re\mathcal{Sh}(\mathcal{S}_-)$.

By simply Booleanizing Theorem 1.4.5, we have the following result.

Theorem 2.3. Let φ: $(\mathcal{S}_+, \mathcal{L}_+) \to (\mathcal{S}_-, \mathcal{L}_-)$ be an X-sketch X-map. Then the X-functors $a_{\mathcal{L}_-} \circ \varphi^*$ and $a_{\mathcal{L}_-} \circ \varphi^* \circ \imath_{\mathcal{L}_+} \circ a_{\mathcal{L}_+}$ from $\mathcal{BP}re\mathcal{Sh}(\mathcal{S}_+)$ to $\mathcal{BSh}(\mathcal{S}_-, \mathcal{L}_-)$ are naturally X-isomorphic.

3. RELATIONS BETWEEN TWO BOOLEAN LIMIT SKETCHES

Let f: $\mathrm{X}_- \to \mathrm{X}_+$ be a morphism of **BLoc**, which shall be fixed throughout this section.

Theorem 3.1. Given small $X_\pm$-sets $\mathscr{V}_\pm$, there is a bijection between the $\mathfrak{f}$-functions from $\mathscr{V}_+$ to $\mathscr{V}_-$ and the X-functions from $\mathfrak{f}^*\mathscr{V}_+$ to $\mathscr{V}_-$.

Proof. It is easy to see that there is a bijection between the $\mathfrak{f}$-functions from $\mathscr{V}_+$ to $\mathscr{V}_-$ and the morphisms from $\underline{\mathfrak{f}}^*\mathscr{V}_+$ to $\mathscr{V}_-$ in the category $\underline{\mathbf{BEns}}$(X). The celebrated adjunction from $\underline{\mathbf{BEns}}$(X) to **BEns**(X) discussed in Nishimura (1995b, Theorem 1.2) gives a bijection

$$\underline{\mathbf{BEns}}(X)(\underline{\mathfrak{f}}^*\mathscr{V}_+, \mathscr{V}_-) \cong \mathbf{BEns}(X)(\mathfrak{f}^*\mathscr{V}_+, \mathscr{V}_-)$$

Therefore the desired conclusion follows. ■

By the same token, we have the following result.

Theorem 3.2. Given small $X_\pm$-categories $\mathscr{C}_\pm$, there is a bijection between the $\mathfrak{f}$-functions from $\mathscr{C}_+$ to $\mathscr{C}_-$ and the X_--functors from $\mathfrak{f}^*\mathscr{C}_+$ to $\mathscr{C}_-$.

The X_--functor corresponding to an $\mathfrak{f}$-functor $\mathscr{F}: \mathscr{C}_+ \to \mathscr{C}_-$ in the above theorem is denoted by $\mathscr{F}_{X_-}$ while the $\mathfrak{f}$-functor corresponding to an X_--functor $\mathscr{G}: \mathfrak{f}^*\mathscr{C}_+ \to \mathscr{C}_-$ under the above theorem is denoted by $\mathscr{G}_\mathfrak{f}$.

It is easy to see the following result.

Lemma 3.3. For any $\mathfrak{f}$-functor $\mathscr{F}: \mathscr{C}_+ \to \mathscr{C}_-$, any X_+-functor $\mathscr{H}: \mathscr{D}_+ \to \mathscr{C}_+$, and any X_--functor $\mathscr{K}: \mathscr{C}_- \to \mathscr{D}_-$, we have $(\mathscr{K} \circ \mathscr{F} \circ \mathscr{H})_{X_-} = \mathscr{K} \circ \mathscr{F}_{X_-} \circ \mathfrak{f}^*\mathscr{H}$.

Example 3.4. Let $\mathscr{C}_+$ be a small X_+-category. The assignment

$$\mathscr{K} \in \mathrm{Ob}\,\mathscr{BPreSh}(X_+; \mathscr{C}_+) \mapsto \mathfrak{f}^*\mathscr{K} \in \mathrm{Ob}\,\mathscr{BPreSh}(X_-; \mathfrak{f}^*\mathscr{C}_+)$$

naturally induces an $\mathfrak{f}$-functor, which is to be denoted by $\mathfrak{f}^*_{\mathscr{BPreSh}}[\mathscr{C}_+]$. For any $x \in \mathrm{Ob}\,\mathscr{C}$ such that $E\mathscr{K}=Ex$, $(\mathfrak{f}^*_{\mathscr{BPreSh}}[\mathscr{C}_+]\mathscr{K})(\mathfrak{f}^*x) = \mathfrak{f}^*(\mathscr{K}x)$.

Theorem 3.5. In the above example, the $\mathfrak{f}$-functor $\mathfrak{f}^*_{\mathscr{BPreSh}}[\mathscr{C}_+]$ maps small X_+-colimits to X_--colimits and maps small X_+-limits to X_--limits.

Proof. The Booleanization of Schubert (1972, Item 8.5.1) guarantees that X_+-colimits in $\mathscr{BPreSh}(X_+; \mathscr{C}_+)$ and X_--colimits in $\mathscr{BPreSh}(X_-; \mathfrak{f}^*\mathscr{C}_+)$ can be computed componentwise. Since $\mathfrak{f}^*_{\mathscr{BEns}}$ maps small X_+-colimits to X_--colimits, the desired first half of the theorem follows. The remaining half of the theorem can be dealt with similarly.

Theorem 3.6. Let $\mathscr{F}$ be a contravariant $\mathfrak{f}$-functor from a small X_+-category $\mathscr{C}_+$ to a small-X_--complete X_--category $\mathscr{D}_-$. Then there is, up to natural $\mathfrak{f}$-isomorphisms, a unique $\mathfrak{f}$-functor $\mathscr{G}: \mathscr{BPreSh}(X_+; \mathscr{C}_+) \to \mathscr{D}_-$ mapping small X_+-colimits to X_--colimits and making the following diagram commutative:

$$\begin{array}{ccc} \mathscr{BPreSh}(X_+;\mathscr{C}_+) & \xrightarrow{\ \mathscr{G}\ } & \mathscr{D}_- \\ {\scriptstyle y}\uparrow & \nearrow{\scriptstyle \mathscr{F}} & \\ \mathscr{C}_+ & & \end{array}$$

Proof. The uniqueness part is obvious, since every object of $\mathscr{BPreSh}(X_+;\mathscr{C}_+)$ is an X_+-colimit of the image of a small partial X_+-diagram in $\mathscr{C}_+$ under the Yoneda embedding y. By Booleanizing MacLane and Moerdijk (1992), Chapter I, §5, Corollary 4 of Theorem 2, we can see that there is an X_--functor $\mathscr{H}$ preserving small X_--colimits and making the diagram

$$\begin{array}{ccc} \mathscr{BPreSh}(X_-;\mathfrak{f}^*\mathscr{C}_+) & \xrightarrow{\ \mathscr{H}\ } & \mathscr{D}_- \\ {\scriptstyle y}\uparrow & \nearrow{\scriptstyle \mathscr{F}_{X_-}} & \\ \mathfrak{f}^*\mathscr{C}_+ & & \end{array}$$

commutative. The desired $\mathscr{G}$ can be obtained as $\mathscr{H} \circ \mathfrak{f}^*_{\mathscr{BPreSh}}[\mathscr{C}_+]$. ∎

Theorem 3.7. Let $\mathscr{F}$ be an $\mathfrak{f}$-functor from a small X_+-category $\mathscr{C}$ to a small X_--category $\mathscr{C}_-$. Then there is, up to natural $\mathfrak{f}$-isomorphisms, a unique $\mathfrak{f}$-functor

$$\pi^*[\mathscr{F}]: \mathscr{BPreSh}(X_+;\mathscr{C}_+) \to \mathscr{BPreSh}(X_-;\mathscr{C}_-)$$

mapping small X_+-colimits to X_--colimits and making the following diagram commutative:

$$\begin{array}{ccc} \mathscr{BPreSh}(X_+;\mathscr{C}_+) & \xrightarrow{\ \pi^*[\mathscr{F}]\ } & \mathscr{BPreSh}(X_-;\mathscr{C}_-) \\ {\scriptstyle y}\uparrow & & \uparrow{\scriptstyle y} \\ \mathscr{C}_+ & \xrightarrow{\ \mathscr{F}\ } & \mathscr{C}_- \end{array}$$

Proof. Take $\mathscr{BPreSh}(X_-;\mathscr{C}_-)$ for $\mathscr{D}_-$ and $y^* \circ \mathscr{F}: \mathscr{C}_+ \to \mathscr{BPreSh}(X_-,\mathscr{C}_-)$ for $\mathscr{F}: \mathscr{C}_+ \to \mathscr{D}_-$ in the above theorem. ∎

Theorem 3.8. Let $\mathfrak{g}: X_1 \to X_2$ and $\mathfrak{h}: X_2 \to X_3$ be morphisms of **BLoc**. Let $\mathscr{G}: \mathscr{C}_2 \to \mathscr{C}_1$ be a small $\mathfrak{g}$-functor and $\mathscr{H}: \mathscr{C}_3 \to \mathscr{C}_2$ a small $\mathfrak{h}$-functor. Then the $\mathfrak{h} \circ \mathfrak{g}$-functors $\pi^*[\mathscr{G} \circ \mathscr{H}]$ and $\pi^*[\mathscr{G}] \circ \pi^*[\mathscr{H}]$ from $\mathscr{BPreSh}(X_3;\mathscr{C}_3)$ to $\mathscr{BPreSh}(X_1;\mathscr{C}_1)$ are naturally $\mathfrak{h} \circ \mathfrak{g}$-isomorphic.

Proof. Consider the following diagram:

$$\begin{array}{ccccc} \mathscr{BPreSh}(X_3;\mathscr{C}_3) & \xrightarrow{\pi^*[\mathscr{H}]} & \mathscr{BPreSh}(X_2;\mathscr{C}_2) & \xrightarrow{\pi^*[\mathscr{G}]} & \mathscr{BPreSh}(X_1;\mathscr{C}_1)\\ \uparrow y & & \uparrow y & & \uparrow y \\ \mathscr{C}_3 & \xrightarrow[\mathscr{H}]{} & \mathscr{C}_2 & \xrightarrow[\mathscr{G}]{} & \mathscr{C}_1 \end{array}$$

The commutativity of the two inner squares implies the commutativity of the outer rectangle, so that

$$\pi^*[\mathscr{G} \circ \mathscr{H}] \cong_{\ell\circ g} \pi^*[\mathscr{G}] \circ \pi^*[\mathscr{H}]$$

as was desired. ■

Let $(\mathscr{C}_\pm, \mathscr{L}_\pm)$ be $X_\pm$-sketches. An f-functor $\mathscr{F}$: $\mathscr{C}_+ \to \mathscr{C}_-$ is called an *f-sketch f-map* if $\mathscr{F}$ maps every X_+-cone in $\mathscr{L}_+$ into X_--cones in $\mathscr{L}_-$. Each sketch f-map $\mathscr{F}$: $(\mathscr{C}_+,\mathscr{L}_+) \to (\mathscr{C}_-,\mathscr{L}_-)$ gives rise to its associated f-functor

$$a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}] \circ i_{\mathscr{L}_+}: \mathscr{BSh}(X_+; \mathscr{C}_+,\mathscr{L}_+) \to \mathscr{BSh}(X_-; \mathscr{C}_-,\mathscr{L}_-)$$

to be denoted by $\pi^*[\mathscr{F}; (\mathscr{C}_+,\mathscr{L}_+), (\mathscr{S}_-,\mathscr{L}_-)]$ or $\pi^*[\mathscr{F};\mathscr{L}_+,\mathscr{L}_-]$. Theorem 3.2 is a variant of the following.

Theorem 3.9. Given $X_\pm$-sketches $(\mathscr{S}_\pm,\mathscr{L}_\pm)$, an f-functor $\mathscr{F}:\mathscr{S}_+ \to \mathscr{S}_-$ is an f-sketch f-map from $(\mathscr{S}_+,\mathscr{L}_+)$ to $(\mathscr{L}_-,\mathscr{L}_-)$ iff the X_--functor $\mathscr{F}_{X_-}$: $f^*\mathscr{S}_+ \to \mathscr{S}_-$ is an X_--sketch X_--map from $(f^*\mathscr{S}_+, f^*\mathscr{L}_+)$ to $(\mathscr{S}_-,\mathscr{L}_-)$.

Let $\mathscr{F}$: $(\mathscr{C}_+,\mathscr{L}_+) \to (\mathscr{C}_-,\mathscr{L}_-)$ be an f-sketch f-map. By FOTP it is easy to see the following result.

Lemma 3.10. The X_--category $f^*\mathscr{BPreSh}(X_+; \mathscr{C}_+)$ can naturally be put down as an X_--subcategory of X_--category $\mathscr{BPreSh}(X_-; \mathscr{C}_-)$ with a natural injection

$$\jmath[\mathscr{C}_+]: f^*\mathscr{BPreSh}(X_+; \mathscr{C}_+) \to \mathscr{BPreSh}(X_-; \mathscr{C}_-)$$

and the following diagram is commutative up to natural X_--isomorphisms:

$$\begin{array}{ccc} \mathscr{BPreSh}(X_-;\mathscr{C}_-) & \xleftarrow{\pi^*[\mathscr{F}_{X_-}]} & \mathscr{BPreSh}(X_-; f^*\mathscr{C}_+)\\ & \nwarrow^{\pi^*[\mathscr{F}]_{X_-}} & \uparrow \jmath[\mathscr{C}_+] \\ & & f^*\mathscr{BPreSh}(X_+;\mathscr{C}_+) \end{array}$$

The pair $(f^*\mathscr{C}_+, f^*\mathscr{L}_+)$ is an X_--sketch, for which we have the following result.

Lemma 3.11. The X_--category $f^*\mathscr{BSh}(X_+; \mathscr{C}_+, \mathscr{L}_+)$ can naturally be put down as an X_--subcategory of the X_--category $\mathscr{BSh}(X_-; f^*\mathscr{C}_+, f^*\mathscr{L}_+)$ with a natural injection

$$\jmath[\mathscr{C}_+, \mathscr{L}_+]: f^*\mathscr{BSh}(X_+; \mathscr{C}_+, \mathscr{L}_+) \to \mathscr{BSh}(X_-; f^*\mathscr{C}_+, f^*\mathscr{L}_+)$$

and the following diagram is commutative up to natural X_--isomorphisms:

$$\begin{array}{ccc}
\mathscr{BSh}(X_-; f^*\mathscr{C}_+, f^*\mathscr{L}_+) & \xleftarrow{\;a_{f^*\mathscr{L}_+}\;} & \mathscr{BPreSh}(X_-; f^*\mathscr{C}_+) \\
\uparrow \jmath[\mathscr{C}_+, \mathscr{L}_+] & & \uparrow \jmath[\mathscr{C}_+] \\
f^*\mathscr{BSh}(X_+; \mathscr{C}_+, \mathscr{L}_+) & \xleftarrow[\;f^*a_{\mathscr{L}_+}\;]{} & f^*\mathscr{BPreSh}(X_+; \mathscr{C}_+)
\end{array}$$

Proof. We can use a Booleanized version of the colimit construction of Borceux (1994, Vol. 1, Theorem 6.2.5) for computing $a_{\mathscr{L}_+}$ and $a_{f^*\mathscr{L}_+}$. Thus the desired result follows readily from Theorem 3.5. ■

The proof of the above lemma shows also the following result.

Lemma 3.12. The following diagram is commutative up to natural X_--isomorphisms:

$$\begin{array}{ccc}
\mathscr{BPreSh}(X_-; f^*\mathscr{C}_+) & \xleftarrow{\;i_{f^*\mathscr{L}_+}\;} & \mathscr{BSh}(X_-; f^*\mathscr{C}_+, f^*\mathscr{L}_+) \\
\uparrow \jmath[\mathscr{C}_+] & & \uparrow \jmath[\mathscr{C}_+, \mathscr{L}_+] \\
f^*\mathscr{BPreSh}(X_+; \mathscr{C}_+) & \xleftarrow[\;f^*i_{\mathscr{L}_+}\;]{} & f^*\mathscr{BSh}(X_+; \mathscr{C}_+, \mathscr{L}_+)
\end{array}$$

Theorem 3.13. The f-functors $a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}]$ and $a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}] \circ i_{\mathscr{L}_+} \circ a_{\mathscr{L}_+}$ from $\mathscr{BSh}(X_-; \mathscr{C}_-, \mathscr{L}_-)$ is $\mathscr{BSh}(X_-; \mathscr{C}_-, \mathscr{L}_-)$ are naturally f-isomorphic.

Proof. Due to Theorem 3.2, it suffices to show that X_--functors $(a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}])_{X_-}$ and $(a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}] \circ i_{\mathscr{L}_+} \circ a_{\mathscr{L}_+})_{X_-}$ are naturally X_--isomorphic. Due to Lemma 3.3 we have $(a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}])_{X_-} = a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}]_{X_-}$ and

$$(a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}] \circ i_{\mathscr{L}_+} \circ a_{\mathscr{L}_+})_{X_-} = a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}]_{X_-} \circ f^*(a_{\mathscr{L}_+} \circ i_{\mathscr{L}_+})$$
$$= a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}]_{X_-} \circ f^* i_{\mathscr{L}_+} \circ f^* a_{\mathscr{L}_+}$$

Since $\mathscr{F}_{X_-}$: $f^*\mathscr{C}_+ \to \mathscr{C}_-$ is X_--left-exact with $f^*\mathscr{C}_+$ being X_--finitely X_--complete,

$$\pi[\mathscr{F}_{X_-}] = (\pi_*[\mathscr{F}_{X_-}], \pi^*[\mathscr{F}_{X_-}]): \mathscr{BPreSh}(X_-; \mathscr{C}_-) \to \mathscr{BPreSh}(X_-; f^*\mathscr{C}_+)$$

is an X_--geometric morphism. Therefore Theorem 2.3 guarantees that X_--functors $a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}_{X_-}] \circ i_{f^*\mathscr{L}_+} \circ a_{f^*\mathscr{L}_+}$ and $a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}_{X_-}]$ are naturally X_--isomorphic. Therefore we have

$$a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}]_{X_-} \circ f^* i_{\mathscr{L}_+} \circ f^* a_{\mathscr{L}_+}$$
$$\cong_{X_-} a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}_{X_-}] \circ \jmath[\mathscr{C}_+] \circ f^* i_{\mathscr{L}_+} \circ f^* a_{\mathscr{L}_+} \qquad \text{(Lemma 3.10)}$$
$$\cong_{X_-} a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}_{X_-}] \circ i_{f^*\mathscr{L}_+} \circ \jmath[\mathscr{C}_+, \mathscr{L}_+] \circ f^* a_{\mathscr{L}_+} \qquad \text{(Lemma 3.12)}$$
$$\cong_{X_-} a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}_{X_-}] \circ i_{f^*\mathscr{L}_+} \circ a_{f^*\mathscr{L}_+} \circ \jmath[\mathscr{C}_+] \qquad \text{(Lemma 3.11)}$$
$$\cong_{X_-} a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}_{X_-}] \circ \jmath[\mathscr{C}_+] \qquad \text{(Theorem 2.3)}$$
$$\cong_{X_-} a_{\mathscr{L}_-} \circ \pi^*[\mathscr{F}]_{X_-} \qquad \text{(Lemma 3.10)}$$

Thus the desired result follows at once. ■

Theorem 3.14. Let g: $X_1 \to X_2$ and h: $X_2 \to X_3$ be morphisms of **BLoc**. Let $\mathscr{G}$: $(\mathscr{C}_2, \mathscr{L}_2) \to (\mathscr{C}_1, \mathscr{L}_1)$ be a g-sketch g-map and $\mathscr{H}$: $(\mathscr{C}_3, \mathscr{L}_3) \to (\mathscr{C}_2, \mathscr{L}_2)$ be an h-sketch h-map. Then the $h \circ g$ functors $\pi^*[\mathscr{G} \circ \mathscr{H}; \mathscr{L}_3, \mathscr{L}_1]$ and $\pi^*[\mathscr{G}; \mathscr{L}_2, \mathscr{L}_1] \circ \pi^*[\mathscr{H}; \mathscr{L}_3, \mathscr{L}_2]$ from $\mathscr{BSh}(X_3; \mathscr{C}_3, \mathscr{L}_3)$ to $\mathscr{BSh}(X_1; \mathscr{C}_1, \mathscr{L}_1)$ are naturally $h \circ g$-isomorphic.

Proof. It suffices to note that

$$\pi^*[\mathscr{G}; \mathscr{L}_2, \mathscr{L}_1] \circ \pi^*[\mathscr{H}; \mathscr{L}_3, \mathscr{L}_2]$$
$$= a_{\mathscr{L}_1} \circ \pi^*[\mathscr{G}] \circ i_{\mathscr{L}_2} \circ a_{\mathscr{L}_2} \circ \pi^*[\mathscr{H}] \circ i_{\mathscr{L}_3}$$
$$\cong_{h \circ g} a_{\mathscr{L}_1} \circ \pi^*[\mathscr{G}] \circ \pi^*[\mathscr{H}] \circ i_{\mathscr{L}_3} \qquad \text{(Theorem 3.13)}$$
$$\cong_{h \circ g} a_{\mathscr{L}_1} \circ \pi^*[\mathscr{G} \circ \mathscr{H}] \circ i_{\mathscr{L}_3} \qquad \text{(Theorem 3.8)}$$
$$= \pi^*[\mathscr{G} \circ \mathscr{H}; \mathscr{L}_3, \mathscr{L}_1] \quad ■$$

4. QUANTIZED LIMIT SKETCHES

Let us introduce the category to be denoted by **BCat**. Its objects are all pairs $(X, \mathscr{A})$ of a Boolean locale X and a small X-category $\mathscr{A}$. A morphism from $(X, \mathscr{A})$ to $(Y, \mathscr{B})$ in **BCat** is a pair $(f, \mathscr{F})$ of a morphism $f: X \to Y$ in **BLoc** and an f-functor $\mathscr{F}: \mathscr{B} \to \mathscr{A}$. The composition $(g, \mathscr{G}) \circ (f, \mathscr{F})$ of morphisms $(f, \mathscr{F}): (X, \mathscr{A}) \to (Y, \mathscr{B})$ and $(g, \mathscr{G}): (Y, \mathscr{B}) \to (Z, \mathscr{C})$ is defined to be $(g \circ f, \mathscr{F} \circ \mathscr{G})$. The category **BCat** has small coproducts, with respect to which the category **BCat** can and shall be put down as an orthogonal category. The assignments $(X, \mathscr{A}) \in \text{Ob}$ **BCat** $\mapsto X \in \text{Ob}$ **BLoc** and $(f, \mathscr{F}) \in \text{Mor}$ **BCat** $\mapsto f \in \text{Mor}$ **BLoc** constitute a functor to be denoted by $\boldsymbol{\theta}_{\mathbf{BLoc}}$.

We now introduce a category to be denoted by **BObj**. Its objects are all triples $(X, \mathscr{A}, a)$ such that $(X, \mathscr{A}) \in \text{Ob}$ **BCat** and a is a total object of the X-category $\mathscr{A}$. A morphism from $(X, \mathscr{A}, a)$ to $(Y, \mathscr{B}, b)$ in **BObj** is a triple $(f, \mathscr{F}, \mathscr{f})$ such that $(f, \mathscr{F})$ is a morphism from $(X, \mathscr{A})$ to $(Y, \mathscr{B})$ in the category **BCat** and $\mathscr{f}$ is a total morphism from $\mathscr{F}b$ to a in the X-category $\mathscr{A}$. The composition $(g, \mathscr{G}, \mathscr{g}) \circ (f, \mathscr{F}, \mathscr{f})$ of $(f, \mathscr{F}, \mathscr{f}): (X, \mathscr{A}, a) \to (Y, \mathscr{B}, b)$ and $(g, \mathscr{G}, \mathscr{g}): (Y, \mathscr{B}, b) \to (Z, \mathscr{C}, c)$ in **BObj** is defined to be $(g \circ f, \mathscr{F} \circ \mathscr{G}, \mathscr{f} \circ \mathscr{F}\mathscr{g})$. It is easy to see that the category **BObj** has small coproducts, with respect to which **BObj** can and shall be put down as an orthogonal category. The assignments

$$(X, \mathscr{A}, a) \in \text{Ob } \mathbf{BObj} \mapsto (X, \mathscr{A}) \in \text{Ob } \mathbf{BCat}$$

$$(f, \mathscr{F}, \mathscr{f}) \in \text{Ob } \mathbf{BObj} \mapsto (f, \mathscr{F}) \in \text{Mor } \mathbf{BCat}$$

constitute a functor from the category **BObj** to the category **BCat** to be denoted by $\boldsymbol{\theta}_{\mathbf{BCat}}$.

Let $\mathscr{M}$ be a manual of Boolean locales, which shall be fixed throughout the rest of this section. An *empirical framework over* $\mathscr{M}$ is a functor Φ from $\mathscr{M}$ to **BCat** subject to the following conditions:

(4.1) It maps orthogonal $\mathscr{M}$-sum diagrams to orthogonal sum diagrams in **BCat**.

(4.2) $\boldsymbol{\theta}_{\mathbf{BLoc}} \circ \Phi$ is the identity functor of $\mathscr{M}$ into **BLoc**.

For an empirical framework Φ over $\mathscr{M}$, we denote by $\Phi_{\mathscr{Cat}}$ the function with the same domain of Φ such that $\Phi(X) = (X, \Phi_{\mathscr{Cat}}(X))$ for each $X \in \text{Ob}\mathscr{M}$ and $\Phi(f) = (f, \Phi_{\mathscr{Cat}}(f))$ for each $f \in \text{Mor } \mathscr{M}$.

Example 4.1. For each $f: X_- \to X_+ \in \text{Mor } \mathscr{M}$, the assignment

$$(\mathscr{S}, \mathscr{L}) \in \text{Ob}\,\mathscr{BSketch}(X_+) \mapsto (f^*\mathscr{S}, f^*\mathscr{L}) \in \text{Ob}\,\mathscr{BSketch}(X_-)$$

naturally induces an f-functor from $\mathscr{BSketch}(X_+)$ to $\mathscr{BSketch}(X_-)$ to be denoted by $f^*_{\mathscr{BSketch}}$. The assignments $X \in \text{Ob}\mathscr{M} \mapsto (X, \mathscr{BSketch}(X))$ and $f \in \text{Mor}\mathscr{M} \mapsto (f, f^*_{\mathscr{BSketch}})$ constitute an empirical framework over $\mathscr{M}$ to be denoted by $\mathfrak{BSketch}$.

Given an empirical framework Φ over $\mathscr{M}$, we now introduce a category to be denoted by **EObj**(Φ). Its objects are all functors $\mathfrak{F}$ from $\mathscr{M}$ to **BObj** abiding by the following conditions:

(4.3) It maps orthogonal $\mathscr{M}$-sum diagrams in $\mathscr{M}$ to orthogonal sum diagrams in **BObj**.

(4.4) $\boldsymbol{\theta}_{\mathbf{BCat}} \circ \mathfrak{F} = \Phi$.

Given such a functor $\mathfrak{F}\colon \mathscr{M} \to \mathbf{BObj}$, we denote by $\mathfrak{F}_{\mathscr{Obj}}$ the function with the same domain of $\mathfrak{F}$ such that the value of $\mathfrak{F}_{\mathscr{Obj}}(?)$ is the third component of the triple $\mathfrak{F}?$. A morphism from $\mathfrak{F}$ to $\mathfrak{G}$ in **EObj**(Φ) is an assignment ζ to each $X \in \mathrm{Ob}\,\mathscr{M}$ of a total morphism $\zeta_X\colon \mathfrak{F}_{\mathscr{Obj}}(X) \to \mathfrak{F}_{\mathscr{Obj}}(Y)$ satisfying the following condition:

(4.5) The diagram

$$\begin{array}{ccc}
\Phi_{\mathscr{Cat}}(f)(\mathfrak{F}_{\mathscr{Obj}}(Y)) & \xrightarrow{\mathfrak{F}_{\mathscr{Obj}}(f)} & \mathfrak{F}_{\mathscr{Obj}}(X) \\
{\scriptstyle \Phi_{\mathscr{Cat}}(f)(\zeta_Y)}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_X} \\
\Phi_{\mathscr{Cat}}(f)(\mathfrak{G}_{\mathscr{Obj}}(Y)) & \xrightarrow[\mathfrak{G}_{\mathscr{Obj}}(f)]{} & \mathfrak{G}_{\mathscr{Obj}}(X)
\end{array}$$

is commutative for every $f\colon X \to Y \in \mathrm{Mor}\,\mathscr{M}$.

The composition $\eta \circ \zeta$ of morphisms $\zeta\colon \mathfrak{F} \to \mathfrak{G}$ and $\eta\colon \mathfrak{G} \to \mathfrak{H}$ in **EObj**(Φ) is defined to be the assignment $X \in \mathrm{Ob}\,\mathscr{M} \mapsto \eta_X \circ \zeta_X$.

Example 4.2. An object of **EObj**($\mathfrak{BSketch}$) is called an *empirical sketch over* $\mathscr{M}$.

An empirical sketch $\mathfrak{S}$ over $\mathscr{M}$ shall be fixed throughout the rest of this section.

Example 4.3. The assignments $X \in \mathrm{Ob}\,\mathscr{M} \mapsto (X, \mathscr{Mod}_X\, \mathfrak{S}_{\mathscr{Obj}}(X))$ and

$$f\colon X_- \to X_+ \in \mathrm{Mor}\,\mathscr{M} \mapsto (f, \pi^*[\mathfrak{S}_{\mathscr{Obj}}(f)_f; \mathfrak{S}_{\mathscr{Obj}}(X_+), \mathfrak{S}_{\mathscr{Obj}}(X_-)])$$

constitute an empirical framework over $\mathscr{M}$ to be denoted by $\mathfrak{Mod}\ \mathfrak{S}$.

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